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Special Section:
Nonlinear Systems in Geophysics: Past Accomplishments and Future Challenges

Key Points:
- The analytical theory of wind-driven sea is developed
- The theoretical results are supported by numerical studies and experimental facts
- A new numerical algorithm for the kinetic equation is proposed

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Weak-Turbulent Theory of Wind-Driven Sea

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Abstract We present a self-consistent analytic theory of the wind-driven sea—the weak-turbulent theory. The base statement of the theory is that the four-wave resonant interactions play the leading role in the energy balance of the wind-driven sea. We study the exact solution of the stationary wave kinetic equation and the self-similar solutions of wave kinetic equation both in fetch- and duration-limited cases. The theory makes possible to explain the nature of universal power-like energy spectra as well as the power-like dependence of the total wave energy and peak frequency from the fetch and the duration.

1. Introduction

Geophysical phenomena are generally difficult for mathematical modeling due to their complexity. Surface ocean waves, both freely propagating and excited by the wind, are a notable exception. A consistent analytic theory of this phenomenon is well developed. It explains the majority of observed experimental facts through a few empiric characteristics. This lucky opportunity is based on the existence of two small physical parameters. First is the air-to-water density ratio $\epsilon = \rho_a/\rho_w \approx 1.2 \cdot 10^{-3}$, and the second one is the average wave steepness. One should distinguish between the local and the average steepness. Let $\eta(\mathbf{r}, t)$ to be the sea surface elevation. The local steepness is defined as $\mu_{loc} = |\nabla\eta|$. In areas of white capping, this quantity could be arbitrary large. Now let $E = \langle\eta^2\rangle$ be the variance of sea surface elevation while ω_p is circular frequency of the spectral peak. The average steepness is defined as follows:

$$\mu^2 = \frac{E\omega_p^4}{g^2}.$$

The characteristic value of steepness for wind-driven waves in the open seas is $0.05-0.07 \ll 1$ (e.g., Badulin et al., 2007). If $\mu \simeq 0.1$, the waves are sharp; if $\mu \simeq 0.13$, the waves become unstable and ready to be destroyed by ineluctable wave breaking. The other gentle slope waves, sea swells ($\mu < 0.04$), do not decay during a hundred thousands periods and can, for example, cross the globe several times.

The localized wave-breaking events where local steepness μ_{loc} is extremely large cover only a relatively small part of the ocean surface, and the ocean may be assumed "linear in mean." This quasi-linearity holds even in strong storms and hurricanes. It allows us to apply the regular expansion procedure in powers of steepness to the basic equations and to construct a self-consistent solid analytic theory of wind-driven seas. We will call it the *weak-turbulent theory* (WTT; Zakharov et al., 1992; Zakharov, 2010; Zakharov & Filonenko, 1966). Nowadays, this theory is supported by massive numeric simulations. WTT explains the bulk of the experimental facts, accumulated in physical oceanography over decades. The main tool of WTT is the wave kinetic equation (WKE) first derived by Hasselmann, (1962, 1963b, 1963a). It is important to stress the fact that WKE is the limiting case of the quantum kinetic equation (QKE), widely used in theoretical physics since Nordheim (1928).

It is clear now that WTT can be used for the description of very broad class of physical phenomena, including waves in magnetohydrodynamics (Galtier et al., 2000), waves in nonlinear optics (Yousefi, 2017), gravitational waves in the Universe (de Oliveira et al., 2013; Galtier & Nazarenko, 2017), plasma waves (Balk, 2000; Yoon et al., 2016), capillary waves (Pushkarev & Zakharov, 1996; Yulin, 2017; Tran, 2017), and Kelvin waves in super-fluid helium (Lvov & Nazarenko, 2010). Nowadays, WTT is the branch of theoretical physics.

The current paper is predominantly, the review, to place WTT in the perspective. Sections 2, 3, 4 contain basic results of WTT including the Kolmogorov-Zakharov (KZ) solutions that became a classic of today's physics. Self-similar solutions for WKE are presented in section 5 as ones providing insight into features of dynamics and explaining essential experimental findings on sea wave evolution. New unpublished results are given in section 6.

2. Basic Equations

We study the potential flow of an ideal incompressible fluid defined in the domain $-\infty < z < \eta(x,y)$. The velocity field is potential: $\mathbf{v} = \nabla\Phi$. The potential evaluated on the surface is $\Psi = \Phi|_{z=\eta}$. The canonical variables η and Ψ satisfy the Hamiltonian equations (Zakharov & Filonenko, 1966)

$$\frac{\partial \eta}{\partial t} = \frac{\partial H}{\partial \Psi}; \quad \frac{\partial \Psi}{\partial t} = -\frac{\partial H}{\partial \eta} \tag{1}$$

where H is the total energy of the fluid. In the case $\mu \ll 1$, one can expand the Hamiltonian in powers of μ (e.g., Zakharov, 1999)

$$H = H_0 + H_1 + H_2 + \dots; \quad H_0 \sim \mu^2, \quad H_1 \sim \mu^3, \quad H_2 \sim \mu^4. \tag{2}$$

We assume thereafter that the water density $\rho_w = 1$. In this case

$$H_0 = \frac{g}{2}\int \eta^2 \mathrm{d}\mathbf{r} + \frac{1}{2}\int \Psi\hat{K}\Psi \mathrm{d}\mathbf{r}.$$

Here $\hat{K} = \sqrt{-\Delta}$ is a positive operator and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian. Keeping only the first three terms in the equation (2) yields the numerically solvable equation (e.g., Korotkevich et al., 2008; Pushkarev & Zakharov, 1996). We perform the nonsymmetrical Fourier transform

$$\eta(\mathbf{r},t) = \int \eta(\mathbf{k})\exp(i\mathbf{kr})\mathrm{d}\mathbf{k} \quad \Psi(\mathbf{r},t) = \int \Psi(\mathbf{k})\exp(i\mathbf{kr})\mathrm{d}\mathbf{k}.$$

The introduction of normal variables

$$\eta_{\mathbf{k}} = \left(\frac{\omega_{\mathbf{k}}}{2g}\right)^{1/2}(a_{\mathbf{k}} + a^*_{-\mathbf{k}}); \quad \Psi_{\mathbf{k}} = i\left(\frac{g}{2\omega_{\mathbf{k}}}\right)^{1/2}(a_{\mathbf{k}} - a^*_{-\mathbf{k}})$$

yields the Hamiltonian

$$H = \int \omega_{\mathbf{k}} a_{\mathbf{k}} a^*_{\mathbf{k}} \mathrm{d}\mathbf{k} + H_3 + H_4 \dots \tag{3}$$

where H_3, H_4 contain cubic and quartic in $a_{\mathbf{k}}$, $a^*_{\mathbf{k}}$ terms correspondingly. If $\mu \ll 1$, these terms are small with respect to the leading quadratic term in (3). Here $\omega_{\mathbf{k}} = \sqrt{g|\mathbf{k}|}$ is the deep water waves dispersion law. The resulting dynamic equations are

$$\frac{\partial a_{\mathbf{k}}}{\partial t} = i\frac{\delta \tilde{H}}{\delta a^*_{\mathbf{k}}} \quad \tilde{H} = \frac{1}{4\pi^2}H. \tag{4}$$

The main question is which weakly nonlinear process is the dominant one? The dominant processes on deep water are the four-wave interactions (Phillips, 1960). In these interactions wave vectors obey the resonance conditions:

$$\begin{aligned} \omega_0 + \omega_1 &= \omega_3 + \omega_4 \\ \mathbf{k}_0 + \mathbf{k}_1 &= \mathbf{k}_3 + \mathbf{k}_4. \end{aligned} \tag{5}$$

Equation (5) defines the resonant manifold. Such resonant four-vector sets are called quadruplets.

The implementation of the proper canonical transformation $a_{\mathbf{k}} \to b_{\mathbf{k}}$ (e.g., Krasitskii, 1994; Zakharov, 1999) makes possible the elimination of the third-order term H_3 of the Hamiltonian expansion. After a canonical transformation, we obtain

$$H = \int \omega_{\mathbf{k}} b_{\mathbf{k}} b^*_{\mathbf{k}} \mathrm{d}\mathbf{k} + \frac{1}{2}\int T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} b^*_{\mathbf{k}} b^*_{\mathbf{k}_1} b_{\mathbf{k}_2} b_{\mathbf{k}_3} \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\mathrm{d}\mathbf{k}\mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2\mathrm{d}\mathbf{k}_3. \tag{6}$$

The coupling coefficient T is a homogeneous function of the order 3, satisfying the natural symmetry conditions. An explicit formula for $T_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}$ can be found in Geogjaev and Zakharov (2017). The new canonical variable $b_{\mathbf{k}}$ obeys the equation

$$\frac{\partial b_{\mathbf{k}}}{\partial t} = \frac{\delta H}{\delta b^*_{\mathbf{k}}}. \tag{7}$$

Let us stress the difference between equations (4) and (7). Equation (4) has only natural motion constants: energy H and momentum $\mathbf{M} = \int \mathbf{K} a_{\mathbf{k}} a^*_{\mathbf{k}} \mathrm{d}\mathbf{k}$. Equation (7) has an additional motion constant—the wave action

$$N = \int b_{\mathbf{k}} b^*_{\mathbf{k}} \mathrm{d}\mathbf{k}.$$

This integral is approximate since five-wave interactions violate it. Its presence, nevertheless, strongly affects the dynamics and kinematics of the gravity surface waves.

The existence of the wave action integral explains fundamental effect observed in wind-driven seas: the downshift of the spectral maximum to the area of large scales. Equation (7) is known as "the Zakharov equation" (Zakharov, 1966, 1968; Zakharov & Filonenko, 1966). It is widely used both in analytic study and in numeric experiments (e.g., Annenkov & Shrira, 1999, 2004; Saffman & Yuen, 1980).

3. Wave Kinetic Equation

The real sea wave ensemble consists of basic quasi-linear waves and "bounded" or "slave" harmonics. The slave harmonics are as important as the basic ones, their nonlinear interaction makes an important contribution to the "effective Hamiltonian" (6). One can say that the canonical transformation $a_{\mathbf{k}} \to b_{\mathbf{k}}$ "cleans" the sea from the slave harmonics (see Zakharov, 2010). They can be described statistically by introducing the "wavenumber spectrum" $N_{\mathbf{k}}$

$$< b_{\mathbf{k}} b^*_{\mathbf{k}'} >= N_{\mathbf{k}} \delta(\mathbf{k} - \mathbf{k}'). \tag{8}$$

The spectrum $N_{\mathbf{k}}$ obeys the equation

$$\frac{\mathrm{d}N_{\mathbf{k}}}{\mathrm{d}t} = \frac{\partial N_{\mathbf{k}}}{\partial t} + \frac{\partial \omega_{\mathbf{k}}}{\partial \mathbf{k}} \nabla_x N_{\mathbf{k}} = S_{\mathrm{nl}}, \tag{9}$$

$$S_{\mathrm{nl}} = F_{\mathbf{k}} - \Gamma_{\mathbf{k}} N_{\mathbf{k}}, \tag{10}$$

where (see Zakharov & Badulin, 2011, for details)

$$F_{\mathbf{k}} = \pi g^2 \int |T_{0123}|^2 N_1 N_2 N_3 \times \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \delta(\omega_{\mathbf{k}} + \omega_1 - \omega_2 - \omega_3) \mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2 \mathrm{d}\mathbf{k}_3, \tag{11}$$

$$\Gamma_{\mathbf{k}} = \pi g^2 \int |T_{0123}|^2 (N_1 N_2 + N_1 N_3 - N_2 N_3) \times \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \delta(\omega_{\mathbf{k}} + \omega_1 - \omega_2 - \omega_3) \mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2 \mathrm{d}\mathbf{k}_3. \tag{12}$$

Here $F_{\mathbf{k}} > 0$ is the "income term" and $\Gamma_{\mathbf{k}} N_{\mathbf{k}}$ is the "outcome term." In a stable situation, $\Gamma_{\mathbf{k}} > 0$, and the income term can balance the outcome term. $\Gamma_{\mathbf{K}}$ can be treated as the coefficient of wave dissipation due to nonlinear wave-wave interactions.

Equation (9) has many names. It is known as the "energy transfer equation" or the "radiation transfer equation." It is also called the "Hasselmann equation," which is quite reasonable: it was Hasselmann (1962, 1963a, 1963b) who first derived this equation directly from the Euler equation for a free surface ideal fluid.

The equation (9) is often erroneously called the "Boltzmann equation." Hereafter we will call it the WKE. The motivation is the fact that theoretical physics widely uses kinetic equations for distribution functions of quasiparticles. Assuming the quasiparticles are bosons, Hamiltonian (6) generates the following quantum kinetic equation (QKE) (Nordheim, 1928):

$$\begin{aligned} S_{\rm nl} \quad = \quad & \pi g^2 \int_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} |T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}|^2 \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(\omega+\omega_1-\omega_2-\omega_3) \\ & \left[\left(N_{\mathbf{k}} N_{\mathbf{k}_1} N_{\mathbf{k}_2} + N_{\mathbf{k}} N_{\mathbf{k}_1} N_{\mathbf{k}_4} - N_{\mathbf{k}} N_{\mathbf{k}_2} N_{\mathbf{k}_3} - N_{\mathbf{k}_1} N_{\mathbf{k}_2} N_{\mathbf{k}_3} \right) + \right. \\ & \left. + \frac{\hbar}{\rho_w g} \left(N_{\mathbf{k}} N_{\mathbf{k}_1} - N_{\mathbf{k}_2} N_{\mathbf{k}_3} \right) \right] \mathrm{d}\mathbf{k}_1 \, \mathrm{d}\mathbf{k}_2 \, \mathrm{d}\mathbf{k}_3 . \end{aligned} \tag{13}$$

Here $\hbar \simeq 1 \times 10^{-34}$Kg·m^2/s is the Plank constant. The quadratic terms in (13) can be neglected for ocean waves with great accuracy, and the QKE turns into the WKE. In the opposite case, when $N = \int N_{\mathbf{k}} \, \mathrm{d}\mathbf{k} \ll \hbar/\rho_w g$, one can neglect the cubic terms and reduce the QKE to the classical Boltzmann equation. That approach can be performed for the surface of liquid helium.

The QKE is widely used in condensed matter physics; the Boltzmann kinetic equation is the main tool in the gas kinetic theory. Their properties bear significant similarities due to standard conservation of motion constants—energy, momentum, and number of particles. They also have fundamental thermodynamic equilibrium solutions, localized in $\mathbf{k}$ space and nonequilibrium solutions that do not deviate too far from the equilibrium ones (see Lifshitz & Pitaevskii, 1981). The WKE is an object of quite different nature. Nevertheless, it is the backbone of the WTT that became a branch of theoretical physics.

4. KZ Solution of WKE

Let us consider the stationary homogeneous WKE

$$S_{\rm nl} = 0. \tag{14}$$

It has thermodynamic equilibrium—the Rayleigh-Jeans solution

$$N_{\mathbf{k}} = \frac{T}{\omega_k + \upsilon} \tag{15}$$

where T and υ are the temperature and chemical potential, respectively. After plugging (15) into (14), one finds the divergence of the income and outcome terms. More accurate analysis shows that the thermodynamic solution is just a formal one, which is completely useless for applications.

But the stationary WKE (14) has other, much more important KZ solutions, which are similar to the Kolmogorov spectra in the theory of turbulence. They are governed by the fluxes of the motion constants, that is, energy, momentum, and wave action. The candidates for motion constants are the following: wave action density

$$N = \int N_{\mathbf{k}} \, \mathrm{d}\mathbf{k},$$

wave energy density

$$E = \int \omega_{\mathbf{k}} N_{\mathbf{k}} \, \mathrm{d}\mathbf{k},$$

and wave momentum density

$$\mathbf{M} = \int \mathbf{k} N_{\mathbf{k}} \, \mathrm{d}\mathbf{k}.$$

Are they really conserved? Conservation of the wave energy means that

$$\int \omega_{\mathbf{k}} S_{\rm nl} \, \mathrm{d}\mathbf{k} = 0. \tag{16}$$

The validity of this relation seems to be an obvious fact. To prove it, one just needs to change the order of integration in (16). But, *the devil is in the details*—the permutation of integration orders in the WKE is not permitted (see Pushkarev et al., 2003, for details). As the result, the integrals N, E, and $\mathbf{M}$ are not the motion constants in a general situation. One can "save" the conservation of the wave action N, assuming that

$|N_{\mathbf{k}}| < C/k^{\alpha}$, $\alpha > 23/6$ in the initial moment of time. The wave energy and momentum "leak," however, to the area of small scales for any initial data.

Let us introduce polar coordinates:

$$k_x = \frac{\omega^2}{g}\cos\theta; \quad k_y = \frac{\omega^2}{g}\sin\theta.$$

Energy, momentum and wave action are the following:

$$\varepsilon(\omega,\theta) = \frac{2\omega^4}{g^2}N(\mathbf{k}); \quad M_x(\omega,\theta) = \frac{2\omega^5\cos\theta}{g^8}N(\mathbf{k}); \quad N(\omega,\theta) = \frac{2\omega^3}{g^2}N(\mathbf{k}).$$

Integration in angle leads to the "one-dimensional" spectra

$$\varepsilon(\omega) = \int_0^{2\pi}\varepsilon(\omega,\theta)\,\mathrm{d}\theta; \quad N(\omega) = \int_0^{2\pi}N(\omega,\theta)\,\mathrm{d}\theta; \quad M_x(\omega) = \int_0^{2\pi}M_x(\omega,\theta)\,\mathrm{d}\theta. \tag{17}$$

Conservation laws for the WKE may be rewritten in the differential form

$$\frac{\partial N(\omega)}{\partial t} = \frac{\partial Q(\omega)}{\partial\omega}; \quad \frac{\partial\varepsilon(\omega)}{\partial t} = -\frac{\partial P(\omega)}{\partial\omega}; \quad \frac{\partial M_x(\omega)}{\partial t} = -\frac{\partial K(\omega)}{\partial\omega}.$$

Here $Q(\omega)$, $P(\omega)$, and $K(\omega)$ have the meaning of the motion constants fluxes. They do not vanish at infinity in the general case, that is, $Q(\infty) = Q > 0$; $P(\infty) = P > 0$; $K(\infty) = K > 0$.

It was shown in Zakharov (2010) that the WKE can be presented in the form

$$\frac{\partial N(\omega,\theta)}{\partial t} = L\hat{A}[N].$$

Here

$$L = \frac{1}{2}\frac{\partial^2}{\partial\omega^2} + \frac{1}{\omega^2}\frac{\partial^2}{\partial\theta^2},$$

while $\hat{A}[N] = L^{-1}S_{\mathrm{nl}}$ is the bounded positive nonlinear operator. The explicit expression for $\hat{A}$ can be found in Zakharov (2010).

Let us introduce

$$A(\omega) = \frac{1}{2\pi}\int_0^{2\pi}\hat{A}(\omega,\theta)\,\mathrm{d}\theta; \quad B(\omega) = \frac{1}{2\pi}\int_0^{2\pi}\hat{A}(\omega,\theta)\cos\theta\,\mathrm{d}\theta.$$

Then the fluxes are

$$Q(\omega) = \frac{\partial A}{\partial\omega}; \quad P(\omega) = -\omega\frac{\partial A}{\partial\omega} - A; \quad K(\omega) = \frac{\omega}{g}\left(\frac{\partial B}{\partial\omega} - 2B\right).$$

The following substitution

$$\hat{A}(\omega,\theta) = \omega Q + P + \frac{2Kg\cos\theta}{\omega}$$

generates exact solutions of equation (12) because

$$L\left(\omega Q + P + \frac{2Kg\cos\theta}{\omega}\right) = 0$$

if Q, P, and K are constants. This three-parameter family comprises the KZ solutions. Q, P, and K are the fluxes of the motion constants. For this case, they are independent of the frequency ω. The simplest and best known solutions appear if $Q = 0, K = 0$

$$E(\omega) = \frac{4\pi c_p g^{4/3}P^{1/3}}{\omega^4}. \tag{18}$$

Here c_p is the "first Kolmogorov constant." This solution (18) was discovered analytically in Zakharov and Filonenko (1966). The physical meaning of this solution is the assumption that some energy source of

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intensity P is present at low frequencies. The WKE governs the transfer of this energy to small wavelength (high frequency) region. This is the "direct cascade," which is quite similar to the classic Kolmogorov spectrum in the incompressible fluid turbulence theory. Such spectra need to be truncated at a certain frequency $\omega \sim \omega_p$. They have a finite total energy E_{tot} and some characteristic steepness μ. Any spectrum of this type creates a "KZ tail" (18) after some time, where

$$P = \alpha \mu^4 \omega_p E_{\text{tot}}.$$

Here α is a constant depending on the shape of the spectrum; μ is the steepness. These ω^{-4} spectra (18) are routinely observed both in wave tank experiments and in the ocean since at least 1971 (Forristall et al., 1978; Liu, 1971). Sometimes they are called "Toba spectra" (Toba, 1972, 1973a, 1973b).

Phillips (1985) offered the another expression for the spectral tail

$$E(\omega) = \frac{\alpha g^2}{\omega^5}. \tag{19}$$

Here $\alpha \simeq 0.0081$ is the universal Phillips' constant. But it was Phillips himself who claimed that his spectrum is not more tenable and asserted that the Zakharov and Filonenko (1966) spectrum is routinely observed in the range just behind the spectral peak. This bold statement does not diminish importance of the Phillips spectrum. This asymptotics is observed for the utmost high frequency area $\omega > (4 \div 5)\omega_p$, while in the energy capacity spectral band, ZF spectrum (18) is realized (e.g., Forristall, 1981, Liu, 1989, Phillips, 1985, Wang & Hwang, 2004) for the frequency band:

$$\omega_p < \omega < 5\omega_p.$$

The Phillips spectrum describes shorter waves: the so-called Phillips sea (see Newell & Zakharov, 2008).

Another important KZ solution appears if one assumes $P = 0, K = 0, Q > 0$:

$$\varepsilon(\omega) = \frac{4\pi c_q Q^{1/3}}{\omega^{11/3}}. \tag{20}$$

This solution describes an inverse cascade of wave action. It implies that there is a source of wave action at high frequencies and a sink at $\omega = 0$. In reality, this can be realized when the waves are excited at small scales. This is the "second KZ solution," similar to the inverse cascade of the wave energy in the theory of 2-D incompressible fluid turbulence. The solutions (18) and (20) are isotropic. The most generic isotropic solution appears if

$$A = P + \omega Q.$$

Now

$$\varepsilon(\omega) = \frac{4\pi c_p}{\omega^4} P^{1/3} F\left(\frac{Q\omega}{P}\right),$$

where $F(\xi)$ is an unknown positive function.

$$F(\xi) \to 1, \quad \xi \to 0; \quad F(\xi) \to \frac{c_q}{c_p}\xi^{1/3}, \quad \xi \to \infty.$$

The most general anisotropic KZ solution has the following form

$$\varepsilon(\omega, \theta) = \frac{P^{1/3}}{\omega^4} R\left(\frac{\omega Q}{P}, \frac{gk}{\omega P}, \theta\right).$$

Here R is a still unknown function, which can be found in the framework of the most simple diffusive model of wave-wave interactions. Zakharov and Pushkarev (1999) suggested accepting

$$A = \frac{\alpha \omega^{15} N^3}{g^4},$$

where α is a tunable constant. Then

$$\varepsilon(\omega,\theta) == \frac{g^{4/3}}{\alpha^{1/3}}\frac{P^{1/3}}{\omega^4}\left(1+\frac{\omega Q}{P}+\frac{2kg}{\omega P}\cos\theta\right)^{1/3}.$$

Note that this solution is not positively defined when $\omega \to \infty$.

One important class of KZ solutions is the asymptotic KZ solutions for direct energy and momentum cascades

$$\varepsilon(\omega,\theta) = \frac{P^{1/3}}{\omega^4} S\left(\frac{kg}{\omega P},\theta\right).$$

Kats and Kontorovich (1974) and Kats et al. (1975) have shown that in the limit of small anisotropy

$$S \to 1 + c\frac{gk\cos\theta}{\omega P} + \ldots .$$

5. Self-Similar Solutions of WKE

Let us consider the homogeneous WKE

$$\frac{\partial\varepsilon}{\partial t} = S_{\rm nl}. \tag{21}$$

One can look for its solutions of the form (see Badulin et al., 2002, 2005; Zakharov, 2005, for details)

$$\varepsilon(\omega,\theta) = a t^{p+q} F(\xi,\theta); \quad \xi = b\omega t^q. \tag{22}$$

Here a, b, p, and q are constants; $F(\xi,\theta)$ is some function of two variables. Homogeneity properties of the collision integral for deep water waves dictate (e.g., Zakharov, 2010)

$$S_{\rm nl} \simeq \omega\varepsilon\left(\frac{\varepsilon\omega^5}{g^2}\right)^2 \simeq \omega\varepsilon\mu^4.$$

By plugging (22) to (21), one finds that the constants are connected by two relations

$$9q = 2p + 1; \quad a = b^{9/2}.$$

Function $F(\xi,\theta)$ obeys the equation

$$\frac{11q-1}{2}F + q\xi F_\xi = S_{\rm nl}.$$

Let us define

$$A = \int \frac{F}{\xi}\,{\rm d}\xi\,{\rm d}\theta; \quad B = \int F\,{\rm d}\xi\,{\rm d}\theta; \quad C = \int \xi\cos\theta F\,{\rm d}\xi\,{\rm d}\theta.$$

Then, the following relations hold

$$N = b^{9/2} t^{(11q-1)/2} A; \quad E = b^{7/2} t^{(9q-1)/2} B; \quad M = \frac{b^{5/2}}{g} t^{(7q-1)/2} C. \tag{23}$$

The flux of wave action

$$Q = \frac{\partial N}{\partial t} = \frac{11q-1}{2} b^{9/2} t^{(11q-3)/2} B.$$

If $Q \neq 0$, the tail of the solution is $\varepsilon \simeq Q^{4/3}/\omega^{11/3}$. The spectrum must be positive, hence $Q \geq 0$. The marginal case $Q = 0$ is realized for $q = 1/11$. In this case

$$\varepsilon(\omega,\theta,t) = b^{9/2} t^{-1/11} F\left(b\omega t^{-1/11},\theta\right). \tag{24}$$

This self-similar solution describes the evolution of the swell. This was studied in detail by Badulin and Zakharov (2017). As far as $Q = 0$, this solution is described by the KZ spectrum (20). If $q > 1/11$, $Q > 0$, then the self-similar solution exhibits the wave action source presence at $\omega \to \infty$.

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If $q = 3/11, Q = \text{const}$, then

$$N \simeq t; \quad E \simeq t^{8/11}; \quad M \simeq t^{5/11}.$$

This is the regime when the wave action grows proportionally to time.

One more important case is realized if the wave energy grows proportionally to time. Now $q = 1/3, p = 1$, $E \sim t$

$$q = \frac{1}{3}; \quad P = \text{const}; \quad E \sim t; \quad M \simeq t^{2/3}.$$

In this case the characteristic wave height $H \simeq t^{1/2}$, while the characteristic period $T \simeq t^{1/3}$, and we get the famous "Toba law"

$$H \simeq T^{3/2}.$$

One more interesting case is $q = 3/7, p = 10/7$. Here $M \sim t$, that is, the momentum grows proportionally to time.

Now we discuss the self-similar solutions of the stationary inhomogeneous WKE, that is, the fetch-limited scenario of wave evolution

$$\frac{2g}{\omega} \cos\theta \frac{\partial \epsilon}{\partial x} = S_{\text{nl}}.$$

The self-similar solution has the following form

$$\epsilon(\omega, \theta, t) = B^5 x^{p+q} G(\zeta, \theta); \quad \zeta = B\omega x^q,$$

where p and q are connected by the magic relation

$$10q = 2p + 1.$$

The function G satisfies the equation

$$\frac{2}{\zeta}\left[\left(5q - \frac{1}{2}\right) F + q\zeta \frac{\partial F}{\partial \zeta}\right] = S_{\text{nl}}.$$

Self-similar solution exists for any $q \geq 1/12$. The marginal case $q = 1/12$ describes a stationary inhomogeneous swell (Badulin & Zakharov, 2017). Energy in this case decays as $t^{-1/12}$. The Toba law $H \simeq T^{3/2}$ is realized if $q = 1/4, p = 3/4$.

Summarizing the previous results, one can assert that the "free" WKE both in stationary ("fetch-limited") and in homogeneous ("duration-limited") cases has a two-parametric family of self-similar solutions.

Let us now study the forced homogeneous WKE

$$\frac{\partial \epsilon}{\partial t} = S_{\text{nl}} + \gamma \omega^{s+1} f(\theta) \epsilon \tag{25}$$

Here $f(\theta)$ is a normalized function of the angle, $2\pi^{1/2} \int_0^{2\pi} f(\theta) \mathrm{d}\theta = 1$, and γ is a constant. Again, one can look for a solution of the form

$$\epsilon(\omega, \theta, t) = \beta^{9/2} t^{(11q-1)/2} F(b\omega t^q, \theta). \tag{26}$$

Plugging (26) to (25), we find that such a solution exists if

$$q = \frac{1}{s+1}; \quad b = \gamma^{1/(1+s)}.$$

Function F satisfies the equation

$$\frac{11q-1}{2} F + q\xi F_\xi = S_{\text{nl}} + \xi^{s+1} f(\theta) F; \quad q = \frac{1}{s+1}; \quad b = \gamma^{1/(s+1)}.$$

Thus, the forced WKE has only one self-similar solution. A similar statement is valid for stationary self-similar solutions. In the stationary fetch-limited case

$$q = \frac{1}{s+2} \quad b = \gamma^{1/(s+2)}.$$

The self-similar solution of the WKE in the homogeneous ocean gives a characteristic frequency (we use the spectral peak frequency ω_p below)

$$\omega_p = \frac{E}{N} \sim t^{-q}; \quad q \geq 1/11,$$

hence, ω_p decreases with time. In the same way, in the stationary ocean

$$\omega_p = \frac{E}{N} \sim \chi^{-q}; \quad q \geq 1/12,$$

the mean frequency decreases with fetch. This effect, which is known as downshift of the spectral peak, is a universal phenomenon. Ocean waves have the tendency to increase their lengths (periods). This effect is the most spectacular demonstration of the fact that the major process on the ocean surface is the four-wave interaction, obeying the resonant conditions (5). Four-wave interaction preserves the wave action. The presence of two motion constants forbids transport of energy only in one direction. A leakage of a small amount of energy to the domain of small scales leads to the formation of an inverse cascade of energy to the large scales.

A quite similar situation is realized in the theory of turbulence in a 2-D incompressible fluid. The presence of two motion integrals, energy and enstrophy, causes leakage of enstrophy to small scales and downshift of wave energy to large scales. This is the reason why large-scale structures like anticyclones are formed in the atmosphere.

6. Numerical Modeling of Swell

The WKE ((8)–(10)) has been a subject for numerical simulation since the beginning of 1970s. Algorithms and codes for its solution were elaborated by a number of research teams (e.g., Gagnaire-Renou et al., 2010; Hasselmann & Hasselmann, 1981; Komatsu & Masuda, 1996; Lavrenov, 2003; Masuda, 1986; Tracy & Resio, 1982; Webb, 1978). In this paper we present the results of a numerical simulation of the WKE obtained using two program codes. The first one is the well-known Webb-Resio-Tracy code (Tracy & Resio, 1982), modified by our group. The second one is a new code developed by Geogjaev and Zakharov (2017). The latter is based on direct integration over quadruplets, satisfying the resonant conditions (5). A detailed description of the code will be published soon.

For comparison with theoretical results, we need the asymptotic behavior of the coupling coefficient $T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ in the case $|\mathbf{k}_1| \simeq |\mathbf{k}_3| \ll |\mathbf{k}_2| \simeq |\mathbf{k}_3|$.

$$T_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} = \frac{1}{2} k_1^2 k_2 T_{\theta_1,\theta_3} + o(k_1^2). \tag{27}$$

For power-law distribution of the wave action

$$N = ak^{-x}$$

the collision integral $S_{\rm nl}$ can be presented in the following form (e.g., Geogjaev & Zakharov, 2017; Zakharov, 2010)

$$S_{\rm nl} = a^3 g^{\frac{3}{2}} k^{-3x+\frac{19}{2}} F_{\rm pow}(x)$$

where $F_{\rm pow}$ is a dimensionless function depending of x only. This function calculated numerically (Geogjaev & Zakharov, 2017) is shown in Figure 1. Using the asymptotics (27), one can find that the integrals in (11) and (12) converge if

$$5/2 < x < 19/4,$$

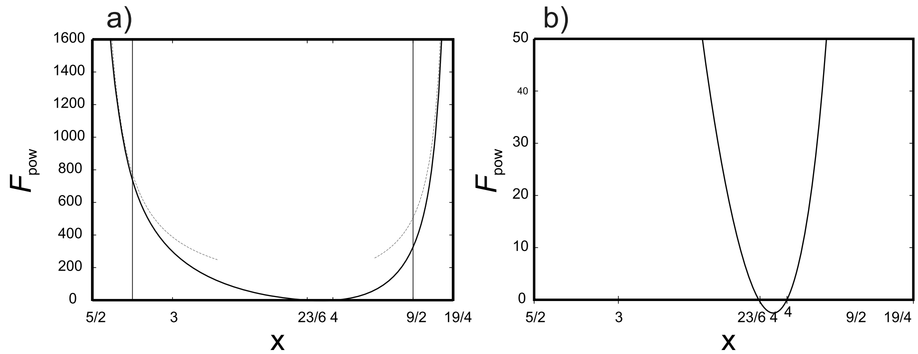

Figure 1. (a) F_{pow} function graph with its hyperbolic asymptotes. (b) The closeup of the function zeroes.

which is the domain of F_{pow}. According to the general theory (Zakharov et al., 1992) the function F_{pow} has exactly two zeroes $x = 4$ and $x = 23/6$. Corresponding KZ spectra have been written above (see (18) and (20)). Spectral fluxes P_0 (energy flux) and Q_0 (wave action flux) can be derived analytically. Thus, the dimensionless constants c_p and c_q can be expressed in terms of the first derivatives of function F_{pow}:

$$c_p = \left(\frac{3}{2\pi F'_{pow}(4)}\right)^{1/3}; \quad c_q = \left(\frac{3}{2\pi F'_{pow}(23/6)}\right)^{1/3}.$$

Our numerical calculation gives

$$c_p = 0.203; \quad c_q = 0.194.$$

Now we present the results of the isotropic swell modeling with the Geogjaev code. The initial energy spectrum shape

$$\varepsilon_0(\omega) = \begin{cases} 2\,\mathrm{m}^2 \cdot \mathrm{s}, & 0.2\pi < \omega \cdot 0.4\pi \\ 0, & \text{elsewhere} \end{cases}$$

is presented in Figure 2a in logarithmic coordinates. The total wave action $N = 1.44\,\mathrm{m}^2\mathrm{s}$ is the constant of motion. Mean root square initial wave height $H_{\mathrm{rms}} = \sqrt{E} \approx 1.13\,\mathrm{m}$ and mean wave period characteristic period $T_{\mathrm{m}} = 2\pi N/E \simeq 7\,\mathrm{s}$ give initial squared steepness $\mu = H_{\mathrm{rms}} k_{\mathrm{m}} \simeq 0.091$ (k_{m} corresponds to the mean period T_{m}). Then we simulate the swell evolution for the time duration of 2×10^6 s or more than 3 weeks. The simulation was performed for the isotropic energy spectrum with 141 point grid. A set of 12,288 quadruplets was used to calculate the WKE collision integral. The parametric diagnostic tail ω^{-4} was applied for frequencies higher than the upper bound of the discrete spectral grid. Variable time step of the WKE integration was 1 s in the beginning and exceeded 1 hr at final stage of the spectrum evolution. As expected, the strict constant of motion N remains constant within the numerical approach.

During the first quite short stage $t \lesssim 3,300\,\mathrm{s}$ (less than 1 day, see Figure 2a), the wave energy remains almost constant as well as the initial spectral shape. Then the Zakharov-Filonenko spectrum $\varepsilon_\omega \sim \omega^{-4}$ forms, and

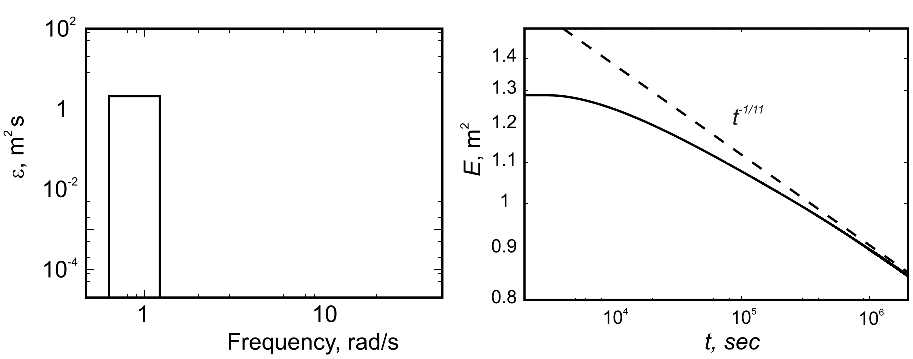

Figure 2. (a) The initial model spectrum $\varepsilon(\omega)$; (b) Evolution of total energy at large times (logarithmic axes).

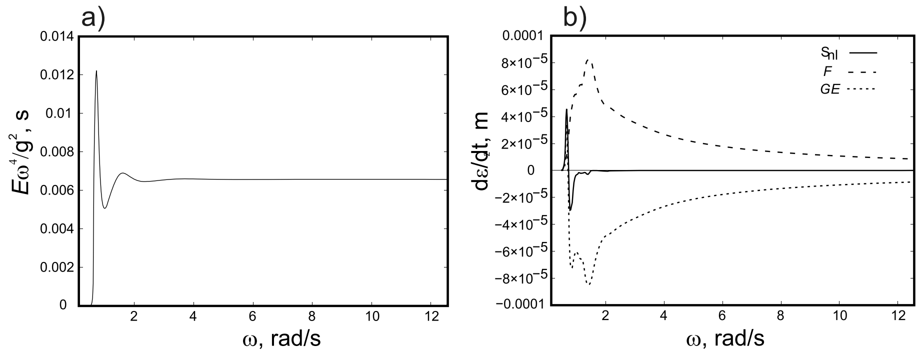

Figure 3. (a) The compensated spectrum $\varepsilon\omega^4/g^2$ at $t = 50,000$ s; (b) collision integral $S_{\rm nl}$ being split into $F_{\mathbf{K}}$ and $\Gamma_{\mathbf{K}} E_{\mathbf{K}}$ (see equations (11) and (12)) at $t = 50,000$ s.

the energy begins to decay. During the second transitional stage $3,300\,\mathrm{s} < t < 50,000\,\mathrm{s}$ (Figure 2a), the spectrum keeps its universal form near the spectral peak, while in the high frequency band, the solution continues to tend to the Zakharov-Filonenko spectrum. For larger time, the spectrum shows pronounced features of self-similarity. The wave action is preserved, and the energy decays like $t^{-1/11}$ (see Figure 2b and equation (24)). The mean (spectral peak) frequency also decays like $t^{-1/11}$ as well as the mean steepness follow the corresponding power-like asymptotics

$$\mu \sim t^{-5/11}.$$

Notice that both observed times $T_1 = 3,300$ s and $T_2 = 50,000$ s can be compared with the characteristic time of the four-wave nonlinear interactions (Zakharov & Badulin, 2011)

$$T_c \simeq \frac{1}{\omega_p \mu^4} \simeq 10^4 \mathrm{s}.$$

Figure 3a presents the compensated spectrum $\varepsilon(\omega)\omega^4/g^2$ at $t = 50,000$ s. One can see that the ω^{-4} asymptotics are perfectly realized.

The two panels of Figure 3 are seminally important. They are made at $t = 50,000$ s. Figure 3a shows quite good correspondence of the numerical solution to the direct cascade solution of Zakharov and Filonenko (1966). Figure 3b presents $S_{\rm nl}$ together with its split to the income and outcome terms. It is clearly seen that $S_{\rm nl} \to 0$ for $\omega > \omega_p$. But their income and outcome components are not small asymptotically. They are large but almost compensate each other.

In the next series of numeric experiments, we have studied the evolution of anisotropic swell (Badulin & Zakharov, 2017). As it was expected, we observed the formation of the self-similar solution with the Zakharov and Filonenko (1966) asymptotics.

7. Numerical Simulation of Wind-Driven Waves

To apply the free kinetic equations (9) and (10) to the real ocean, one should supplement it with source terms

$$\frac{\partial N}{\partial t} = S_{\rm nl} + S_{\rm source}.$$

The source term consists of two components

$$S_{\rm source} = S_{\rm in} + S_{\rm diss}$$

The term $S_{\rm in}$ describes the wave generation by the wind. Excluding catastrophic events like earthquakes, landslides, and so forth, the wind remains the single cause of the ocean surface wave excitation. It is a slow process due to the smallness of the air-water densities ratio. Waves achieve the saturation during approximately 10 thousand of their periods. The oceanographic community spent a lot of efforts on the construction of the algebraic model for $S_{\rm in}$.

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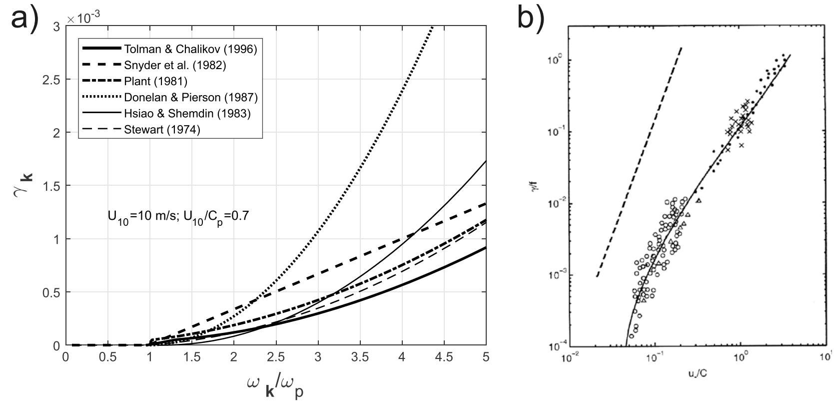

Figure 4. (a) Wind wave growth rates given by different parametric formulas (see legend). (b) Comparison of the experimental data on the wind-induced growth rate $2\pi\gamma_{\text{in}}(\omega)/\omega$ taken from Komen et al. (1995) and the damping due to four-wave interactions, calculated for the narrow in angle spectrum at $\mu = 0.05$ using equation (12) (dashed line).

It is the common belief that waves are generated due to atmospheric boundary layer instability development (e.g., Miles, 1957). It assumes that the corresponding source function S_{source} depends linearly on spectrum

$$S_{\text{in}} = \gamma_{\text{in}}(\omega, \theta)\varepsilon(\omega, \theta). \tag{28}$$

At least dozen of different models for γ_{in} were offered by different authors during the last century (since the seminal papers of Jeffreys (1925, 1926). Figure 4a shows a collection of such dependencies in terms of the dimensionless frequency $\omega_{\mathbf{k}}/\omega_p$ for wind speed at standard height 10 m $U_{10} = 10\,\text{m}\cdot\text{s}^{-1}$ and wave age $U_{10}/C_p = 0.7$. All these dependencies have been obtained experimentally under very different conditions of the turbulent atmospheric boundary layer near the sea surface (the vertical temperature distribution and the distance to the coastline). None of these dependencies were rigorously justified (see the discussion in Pushkarev & Zakharov, 2016). Moreover, the very existence of the universal γ_{in} is not clear.

After including S_{in} in the WKE (10), we obtain

$$S_{\text{nl}} = F_{\mathbf{k}} + (\gamma_{\text{in}} - \Gamma_{\mathbf{k}})N_{\mathbf{k}}. \tag{29}$$

Equation (29) shows that energy input from the wind is compensated by the nonlinear damping $\Gamma_{\mathbf{k}}$. Hence, it is important to compare γ_{in}. It was shown (Zakharov & Badulin, 2011) that for $\omega \gg \omega_p$

$$\frac{\Gamma}{\omega} \simeq C\mu^4 \left(\frac{\omega}{\omega_p}\right)^3.$$

The coefficient C depends on the shape of spectrum. It was shown numerically that it is large. For a narrow angle spectrum, $C > 100$ and, for the isotropic spectra, C is approximately 40% less. As the result, Γ surpasses γ_{in} by at least one order of magnitude (see Figure 4b).

The domination of Γ over γ_{in} leads to the extremely important conclusion:

The main process in the wind-driven sea is the competition of the income and outcome terms in S_{nl}. In fact, typical ocean spectra are the self-similar spectra or combination of those. Parameters of these spectra are defined by the wind interaction.

The second source term S_{diss} describes the loss of the wave energy. Such process takes place in the ocean, for sure, but its role in the total balance of wave energy is overestimated. The most popular model (Komen et al., 1984) for S_{diss} is realized in the WAM-3 wave predicting operational model. In this model

$$S_{\text{diss}} = \gamma_{\text{diss}}\varepsilon(\omega, \theta); \quad \gamma_{\text{diss}} = 1.6\omega\left(\frac{\omega}{\omega_p}\right)^2 \mu^4. \tag{30}$$

We are skeptical about this expression for γ_{diss}. It is neither supported by theory or experimental observations. It is a purely artificial construction, based on strong belief that the evolution of any wind-forced

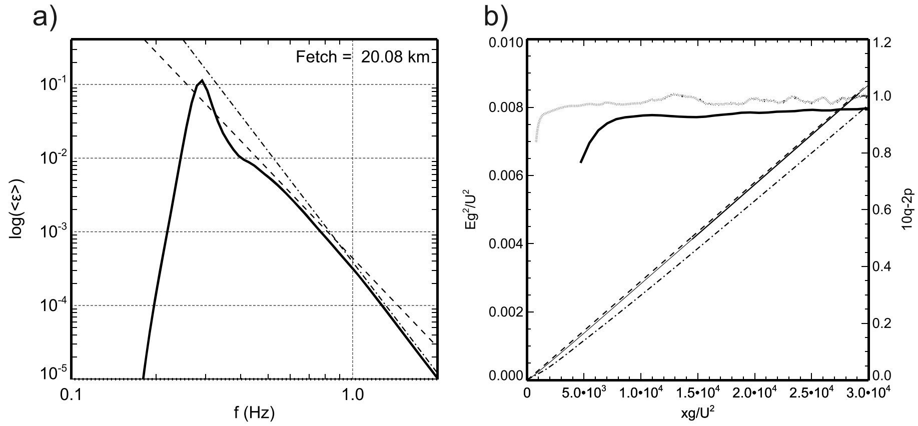

Figure 5. (a) Frequency spectrum for fetch-limited case at $U_{10} = 10$ m/s (solid line), spectrum $\sim f^{-4}$ (dashed line), $\sim f^{-5}$ (dash-dotted line). (b) Dimensionless energy Eg^2/U_{10}^4 versus dimensionless fetch $\chi = xg/U_{10}^2$ (left vertical axis): $U_{10} = 10$ m/s (solid line), $U_{10} = 5$ m/s (dash-dotted line). Self-similar solution with the empirical coefficient $2.9 \cdot 10^{-7} xg/U_{10}^2$ (dashed line). $U_{10} = 10$ m/s (solid line), $U_{10} = 5$ m/s (dashed line). Right vertical axis corresponds to the "magic number" $10q - 2p$ versus the dimensionless fetch χ for fetch-limited case. $U_{10} = 10$ m/s (dotted line), $U_{10} = 5$ m/s (dash-triple-dotted line).

spectrum must end up by the formation of the "fully developed sea." This concept looked obvious back in 1984 (Komen et al., 1984) and now, likely, requires substantial revision. Phase-resolving numerical modeling of JONSWAP spectrum evolution (e.g., Dyachenko et al., 2015; Korotkevich et al., 2008) found that the term (30) can overestimate the dissipation due to the wave breaking by the order of magnitude. Summarizing, we conclude that the instability, driven by the wind input, is arrested not by white capping but by the resonant nonlinear interactions. Numerous white-capping wave breakings form the short-scale Phillips spectrum ω^{-5} (Newell & Zakharov, 2008) and act like a universal sink of energy.

We present below the results of numeric simulation of the stationary WKE.

As an alternative to multiple wind source terms, noncompliant with the set of nonlinear tests (see Pushkarev & Zakharov, 2016), the new ZRP wind-input source term, constructed through the self-similarity analysis of Hasselmann equation, has been proposed (Zakharov et al., 2017):

$$\frac{\gamma}{\omega} = 0.05\frac{\rho_a}{\rho_w}\left(\frac{\omega}{\omega_0}\right)^{4/3}; \quad S_{in}(\omega,\theta) = \gamma(\omega,\theta)\cdot\varepsilon(\omega,\theta), \tag{31}$$

$$\gamma(\omega,\theta) = \begin{cases} 0.05\frac{\rho_a}{\rho_w}\omega\left(\frac{\omega}{\omega_0}\right)^{4/3} q(\theta), \text{for} f_{min} \le f \le f_d, \quad \omega = 2\pi f \\ 0 \quad \text{otherwise,} \end{cases} \tag{32}$$

$$q(\theta) = \begin{cases} \cos 2\theta \quad \text{for} - \pi/4 \le \theta \le \pi/4 \\ 0, \text{otherwise} \end{cases}; \quad \omega_0 = \frac{g}{U_{10}}, \quad \frac{\rho_a}{\rho_w} = 1.3\cdot 10^{-3}, \tag{33}$$

where ρ_a and ρ_w are the air and water density correspondingly. Frequencies f_{min} and f_d depend on the wind speed and should be found empirically: The wind speed at 10-m height was taken as $U_{10} = 10$ m/s and $U_{10} = 5$ m/s, $f_{min} = 0.1$ Hz.

The dissipation was chosen in the "implicit" form: the dynamical part of the spectrum was continued from $f_d = 1.1$ Hz (Resio et al., 2004) by Phillips (1985) spectrum $\varepsilon(f,\theta) \sim \omega^{-5}$, decaying faster than the equilibrium spectrum $\varepsilon(f,\theta) \sim \omega^{-4}$ and providing therefore high-frequency dissipation (Pushkarev & Zakharov, 2016).

This numerical experiment confirmed major predictions of the WTT. Figure 5a shows the formation of KZ spectra. Dimensionless wave energy as function of dimensionless fetch for different wind speeds and corresponding values of the power laws indices as well as the "magic relations" are shown in Figure 5b as the arguments for asymptotic convergence to the predicted value.

8. Experimental Evidence of S_{nl} Domination

In the previous chapter we have shown analytically and numerically that the S_{nl} term dominates over S_{in} term. As S_{diss} in the Hasselmann sea cannot be stronger than γ_{in} (otherwise the waves would not be exited), the term S_{nl} dominates over S_{diss} too. Both the source term and nonlinear wave interaction are the dominating physical processes that take place in a wind-driven sea.

This fact is supported by convincing experimental data, collected in a broad ranges of wind velocities: $3\,\mathrm{m/s} < U_{10} < 30\,\mathrm{m/s}$. Following Kitaigorodskii (1962), we will use hereafter the dimensionless duration and fetch, as well as the dimensionless frequency and energy.

$$\tau = \frac{tg}{U}; \quad \xi = \frac{xg}{U^2}; \quad \sigma = \frac{\omega U}{g}; \quad F = \frac{\epsilon g^2}{U^4}. \tag{34}$$

We also introduce integral dimensionless quantities

$$\tilde{F} = \int_0^\infty F(\sigma)\,\mathrm{d}\sigma; \quad \tilde{\sigma} = \frac{1}{\tilde{F}}\int_0^\infty \sigma F(\sigma)\,\mathrm{d}\sigma; \quad \mu_p = F\sigma. \tag{35}$$

During the last seven decades, many experiments on the measurement of wave energy spectra and their integral characteristics have been performed in wind-driven seas, laboratories, and lakes. The most significant experiments were conducted in the “fetch-dominated frame,” where the sea is stationary in time and the wind blows perpendicular to a coast line. In these challenging and expensive experiments, $\tilde{F}$ and $\tilde{\sigma}$ were measured as functions of the dimensionless fetch only: $\tilde{F} = \tilde{F}(\xi)$, $\tilde{\sigma} = \tilde{\sigma}(\xi)$ following the Kitaigorodskii (1962) similarity approach. Generally, these dependencies are approximated by power-like functions

$$\tilde{F} = \epsilon_0 \xi^p; \quad \tilde{\sigma} = \omega_0 \xi^{-q}. \tag{36}$$

In all the experiments, the exponents p and q vary inside the following ranges:

$$0.7 < p < 1.1; \quad 0.22 < q < 0.33. \tag{37}$$

Let us assume that the observed energy spectra are described by self-similar solutions of the stationary WKE, supplied with the wind pumping increment (31)

$$S_{\mathrm{in}} = m\frac{\rho_{\mathrm{a}}}{\rho_{\mathrm{w}}}\left(\frac{\omega}{\omega_c}\right)^{1+l}\epsilon(\omega,\theta).$$

Here m is some unknown constant. It means that

$$q = \frac{l}{2+l} \quad p = \frac{8-l}{2(2+l)},$$

where p and q are connected by the magic relation

$$q = 2p + 1,$$

and another relation, excluding an indefinite constant b in (23)

$$\epsilon_0^{1/5}\omega_0 = S. \tag{38}$$

This S in (38) is a constant of the order one. It might depend slowly on p.

Results of 23 experiments performed in the open sea and Lake Michigan are presented in Table 1, which represents the majority (but not all) of field experiments collected in physical oceanography for almost half a century (see, for reference, Badulin et al., 2007). Experimental data are compared with the predictions of the analytic theory presented in this paper. According to the theory, the exponents q_χ must coincide with the theoretically predicted value $q_{th} = 2p_x + 1/10$. One can see that the relative difference $z_\chi \simeq \frac{1}{q_\chi}|q_\chi - q_{th}|$ does not exceed 10%. According to the theory, the dimensionless quantity $S = \epsilon^{1/5}\omega_0$ must be a constant of the order one that is supported by Table 1 fairly well. A more accurate theoretical value of S (which is actually a slowly varying function of p) will be presented shortly.

Table 1
Collection of Wind Sea Experiments

	Case	$\epsilon_0 \times 10^7$	p_χ	ω_0	q_χ	q_{th}	α_{ss}	z_χ	S
1.1	Black.Sea	4.410	0.890	15.14	0.28	0.28	0.65	0.010	0.81
1.2	Walsh, U.S. coast (1989)	1.860	1.000	14.45	0.29	0.30	0.30	0.033	0.65
1.3	Kahma and Calkoen (1992) unstable	5.400	0.940	14.20	0.28	0.29	0.59	0.027	0.79
1.4	Kahma and Calkoen (1992) stable	9.300	0.760	12.00	0.24	0.25	0.52	0.040	0.75
1.5	Romero and Melville(2009) stable	9.230	0.740	8.93	0.22	0.25	0.20	0.093	0.55
1.6	Romero and Melville (2009) unstable	5.750	0.810	10.64	0.23	0.26	0.25	0.107	0.60
2.1	Dobson et al. (1989)	12.70	0.750	10.68	0.24	0.25	0.44	0.033	0.71
2.2	Kahma and Pettersson (1994)	5.300	0.930	12.66	0.28	0.29	0.40	0.020	0.70
2.3	Davidan (1980), equations (6) and (8)	4.40	0.84	16.00	0.28	0.30	0.75	0.067	0.86
2.4	JONSWAP no lab (Phillips, 1977)	2.600	1.000	11.18	0.25	0.30	0.16	0.167	0.54
2.5	Kahma and Calkoen (1992) composite	5.200	0.900	13.70	0.27	0.28	0.52	0.033	0.76
2.6	Donelan et al. (1985)	8.410	0.760	11.60	0.23	0.25	0.43	0.073	0.71
2.7	SMB CERC (1977) by Young	7.820	0.840	10.82	0.25	0.27	0.32	0.060	0.65
3.1	Wen et al. (1989)	18.90	0.700	10.40	0.23	0.24	0.53	0.023	0.75
3.2	Evans and Kibblewhite (1990) neutral	2.600	0.872	18.72	0.30	0.27	0.94	−0.085	0.90
3.3	Evans and Kibblewhite (1990) stratified	5.900	0.786	16.27	0.28	0.26	1.05	−0.076	0.92
3.4	Kahma (1981, 1986) rapid growth	3.600	1.000	20.00	0.33	0.30	1.38	−0.100	1.03
3.5	Kahma (1986) average growth	2.000	1.000	22.00	0.33	0.30	1.29	−0.100	1.01
3.6	Donelan et al. (1992)	1.700	1.000	22.62	0.33	0.30	1.27	−0.100	1.00
3.7	Hwang and Wang (2004)	6.191	0.811	11.86	0.24	0.26	0.37	0.084	0.68
3.8	Ross (1978) Michigan unstab	1.200	1.100	11.94	0.27	0.32	0.12	0.167	0.49
3.9	Liu and Ross(1980) stratification	0.680	1.100	12.88	0.27	0.32	0.10	0.167	0.47
3.10	Liu and Ross(1980) Babanin's fit	77.0	0.520	2.36	0.08	0.20	0.01	0.413	0.22
3.11	Davidan (1996), u^* scaling	794.0	1.000	9.16	0.34	0.30	3.74	−0.133	1.39
3.12	Davidan (1996), U_{10} scaling	5.550	0.840	16.34	0.29	0.27	1.00	−0.073	0.92
4.1	JONSWAP (1973)	2.890	1.008	19.72	0.33	0.30	1.14	−0.095	0.97
4.2	Mitsuyasu (1971)	1.600	1.000	21.99	0.33	0.30	1.11	−0.100	0.96

One can supplement the Table 1 with the composite data presented in the monograph by Young (1999) in page 105. This is the result obtained by the author by averaging over many field experiments. According to Young (1999):

$$p_\chi = 0.8; \quad q_\chi = 0.25; \quad \epsilon_0 = 7.5 \cdot 10^{-7}; \quad \omega_0 = 12.56; \quad S = 0.75.$$

Theory predicts $q_{th} = 0.26$, $S \simeq 1$. These data support our statement on the dominance of the nonlinear resonant process over the wind income. Other experimental and numerical data supporting our theory are collected in Badulin et al. (2007). In the field experiments, presented in Table 1, the dimensionless fetch χ varies in the range $10^2 < \chi < 10^5$.

9. Conclusion

We have developed a quite simple theory. The only empiric parameters of the theory are m and l, contained in the shape of the wind-input term. The theory does not consider the problem of the individual wave-breaking event and, therefore, contains no explicit parameterization of white-capping dissipation source term. Instead, we used the implicit dissipation term in the form of the Phillips continuation of the dynamical part of the spectrum, replicating main properties of white-capping dissipation and playing the same role as the viscosity at large Reynolds numbers in the theory of turbulence.

This simplified theory explains the following fundamental facts.

- Universality of energy spectra in the spectral band $\omega_p < \omega < 5\omega_p$. The explanation is simple: The ω^{-4} spectrum is the exact solution of the WKE;

- Power-like dependence of the wave energy and mean frequency of the fetch. The explanation is simple again: The fetch-limited spectra are described by self-similar solutions of WKE.

We think that these results are quite impressive, and we have ambitious plans. We hope that careful study of anisotropic KZ spectra will explain angular spreading of the spectra, increasing with the frequency. And we hope that our new numerical code will help to advance in understanding of wave problems as well as wave forecasting models progress.

Acknowledgments
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